Maimonides in the Guide of the Perplexed I: 73 בנושא שני הקווים on the proposition II: 14 in Apollonius' second book of the conic sections: a synthesis.

Maimonides knew the Chef d'Oeuvre of Apollonius 'The Conics' and he used the proposition II-14 in the formulation of an argument against the Calam at the end of the first part of the *Guide of the perplexed*. He wrote that there are things which are impossible to imagine and however, we observe them or we can demonstrate them. He offered the example of the two lines, the hyperbola and its asymptote which even if produced to infinity, approach nearer one another and come within a distance less than any given distance without meeting. This last point was considered as unimaginable. This proposition puzzled the mathematicians and the philosophers until the beginning of the modern time. Maimonides' geometric argument stimulated the interest of Jewish philosophers and mathematicians. They tried to elaborate, with the help of ancient compositions, independent proves. We find quotations from important gentiles authors related to Apollonius' proposition and bearing the mark of Maimonides' expression. In the present paper we propose a synthesis of these works.
We raise also the issue whether the conics considered by the ancients i.e. the conic sections are equivalent to the conics considered today in analytic geometry.

Maimonides in the Guide of the Perplexed I: 73 בנושא שני הקווים on the proposition II: 14 in Apollonius' second book of the conic sections: a synthesis.

The end of the first part of the Guide of the Perplexes (chapters 71 – 75) is devoted to the refutation of the Calam. The Calam is a pseudo rational and philosophical system developed by the Muslim Doctors in order to contradict the agnostic philosophers of the classical Greek period. Maimonides' aim is contradicting the arguments of the agnostic philosophers and the Muslim Mottecallemin and demonstrating the weakness of their argumentations before expounding his own views.

Maimonides emphasizes the weakness of the arguments of the Calam. For example in chapter 71, he doesn't accept their demonstration of the newness of the world. The thinkers of the Calam insisted on the newness of the world and proved by this way the existence of God. Maimonides however rejects their argumentation and writes that all their pretended proves of the newness of the world are subjected to doubts. 'They are decisive proves only for people who accept fallacious arguments and cannot make the difference between demonstration, dialectic and sophism'. In fact their demonstrations are doubtful and rest on unproven early beginnings.

In chapter 73, Maimonides considers twelve propositions of the Calam. The tenth proposition deals with a fundamental basis of the science of the Calam. The scholars of the Calam, the Mottécallemin, contend that anything imaginable or conceivable is acceptable for the intellect or the faculty of reason. Maimonides seeks to establish the thesis that 'man is not distinguished by having imagination' and that the 'act of imagination is not the act of the intellect but rather the contrary'.

With that aim, Maimonides considers contradictory examples.

1. We can conceive in our imagination a human being with a horse head and wings or other similar creatures. But this is 'false invention' because there isn't such a creature.
2. He insists on the importance of the mathematical sciences and the propositions that we can take from them. 'There are things that men cannot imagine but we can prove that they exist and represent the reality'.
   a. If we consider on the spherical earth two people in two diametrically opposed places. Each of them has the head toward the sky and the feet toward the earth. It is not imaginable that none of them falls. However the facts prove the contrary and each of them considers that it is the other who is upside down.[1]
   b. Another example: 'It has been made clear in the second book of the conic sections[2] that two lines between which there is a certain distance at the outset, may go forth in such a way that the farther they go, this distance diminishes and they come nearer to one another, but without it ever being possible for them to meet even if they are drawn forth to infinity and even though they come nearer to one another

[1] Maimonides does not elaborate. Apparently the reality seems sufficient to prove the absence of pertinence of the imagination. However, for people preceding Newton and the principle of the universal attraction, the reality remained paradoxical. But, surprisingly, the case was not discussed by any commentator.

[2] Literally,ספר החרוטים means the book of the cones. He has in mind Apollonius books of the conics.

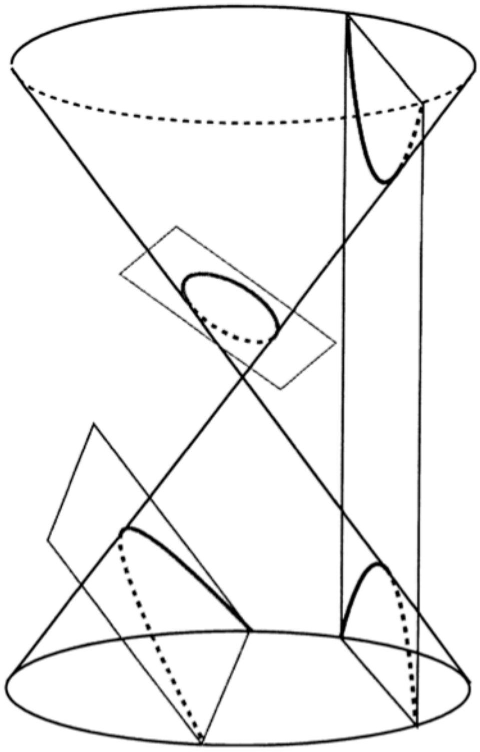

Figure 1: The different types of conic sections, ellipse when the cutting plane cuts only one nappe of the cone, parabola when it is parallel to an element of the cone and hyperbola when it cuts both nappes of the cone. We exclude three particular cases. 1. The plane crosses the vertex: the intersection is a point. 2. The plane is tangent to the cone along an element of the cone: the intersection is a straight line. 3. The plane passes through the axis of the cone: the intersection is two straight lines.

> the farther they go. This cannot be imagined and can in no way enter within the net of imagination. Of these two lines, one is straight and the other curved, as has been made clear there in the above-mentioned work'.[3]

The Hebrew translation of ibn Tibbon[4] was the following:

"וכן התבאר במופת במאמר השני מספר החרוטים יציאת שני קווים יהיה ביניהם בתחילת יציאותם רוחק אחד, וכל אשר ירחקו יסתר הרוחק ההוא ויקרב אחד מהם אל האחד, ולא יתכן הפגשם לעולם ואפילו יצאו לבלתי תכלית. ואע"פ שכל אשר ירחקו יתקרבו. וזה לא יתכן שידומה ולא שיפול בשכבת הדמיון כלל. ושני הקווים ההם, האחד מהם ישר והאחד מעוקם כמו שהתבאר שם".

This is the origin and the context of Maimonides' famous quotation of the proposition II: 14 in Apollonius' book of the conic sections.[5]

Maimonides' quotation and the precise allusion to Apollonius' book of the conic sections prove that Maimonides was well acquainted with this book. It was even adduced that

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[3] Quoted after the translation by Shlomo Pines : Chicago 1963, I. 210.

[4] Samuel ben Judah ibn Tibbon, Provence (Lunel, Arles, Béziers and Marseilles) , about 1160 – 1230. He belonged to a famous family of scholars and translators; he was the son of Judah, the father of Moses and the grand-father of the astronomer Jacob ben Machir.

[5] Apollonius of Perga, celebrated geometer (260 – 200 BCE).

Maimonides is the author of a commentary on Apollonius' conic sections,[6] preserved in an Arab manuscript ascribing it to 'al-ra'is Ibn Imran Musa bin Ubayd Allah al-Israili al-Qurtubi.' This attribution must however be considered with reservation.[7]The text of the proposition II 14 of Apollonius is the following: 'The asymptotes and the section, if produced to infinity, approach nearer one another and come within a distance less than any given distance'.[8]

In Rashed's new edition of Apollonius' treatise of the conic sections, the text of the proposition II 14 is the following: 'Si un point s'éloigne sur une hyperbole H de centre A, il se rapproche autant que l'on veut de l'une des asymptotes sans la rencontrer'.[9]

The demonstration of this proposition is based on the following consideration: Let G and S be two points of the same branch of the hyperbola. Through these two points we draw two parallels: the first passing through G cuts the asymptotes in H and N, the second passing through S cuts the asymptotes in C and L. We assume that SL > SC and GN > GH. We have then, because of the former proposition II 10:

GH . GN = SC . SL = P (1) where P is a constant.

Now because BL > BN, certainly CL > NH (2)

thus SC + SL > GH + GN (3)

Let us prove that **SL > GN**.[10] We draw the straight line BG cutting LC in O. We have AO > AG therefore LO > NG and certainly SL >GN (4)

When point G moves to the right and becomes S, SL > GN and SG= (GH .GN) / SL is smaller than GH.

Therefore when G moves to the right, GH remains > 0 but it becomes always smaller.

GH will become smaller than any quantity ε when G is sufficiently far in the right direction, as soon as GN > P / ε = constant / ε.[11]

The book of Apollonius was translated into Arab during the 9^{th} century and from the 10^{th} century onwards a few important Arab mathematicians published memoirs devoted to some properties of the conic sections and especially to the asymptotic property of the hyperbola considered in the proposition II: 14 of Apollonius. Indeed this proposition puzzled the Arab mathematicians.

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6 See Langermann (1984).

7 See Gad Freudenthal (2000): The Transmission of 'On two lines'p.52, note 3.

8 See Heath (1896).

9 See Rashed (2010) vol 2.

10 Algebric demonstration: We use the *__reductio ad absurdum__* method of reasoning:

1. If SL = GN (4) then SC > GH because of (3)
   and then SC . SL > GH . SL = GH . GN. This is impossible because it contradicts (1).
2. If SL < GN (5) then (3) and (1) give:
   SL + P / SL > GN + P / GN
   0 < GN – SL < P (1 / SL – 1 / GN) = (P / SL.GN) (GN – SL)
   Hence (P / SL.GN) > 1 and P > SL . GN. Now P = GN . GH
   Therefore GN . GH > SL . GN and GH > SL. But GN > SL (5) Thus
   GH + GN = NH > 2 SL > LC because we assumed SL > SC. **This is impossible** because it contradicts (2). Thus SL > GN and therefore SC = P / SL < GH = P / GN.

11 See Rashed (2010), vol 2 pp. 26 – 27.

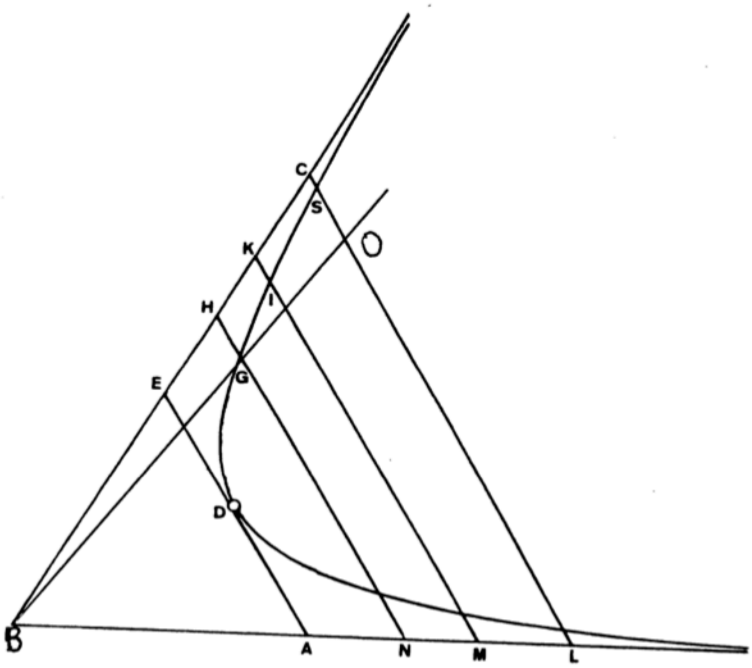

Figure 2: Demonstration of the proposition II-14 of Apollonius. SL > GN and CS < GH. When G moves to the right, the distance GH diminishes and becomes < ε when G is sufficiently far to the right.

Historical demonstrations of the properties of the asymptotes of the hyperbola.

Among these Arab mathematicians we find al-Sijzi[12] (10-11th century), al-Tusi[13] (13th century), al-Qumi (11th century) and al-Haythan (10-11th century). Two Latin manuscripts are also extant; indeed Clagett discovered a Latin manuscript which he edited and entitled: *Tractatus de duabus lineis semper approximantibus sibi invicem et nunquam concurrentibus* or simply *on two lines*.[14] This text is likely the translation of an Arab lost text; however a Hebrew translation is extant in several manuscripts.[15] Both texts were probably used by the authors of the middle age. Clagett studied a second Latin text giving an original demonstration of the asymptotic properties of the asymptotes of the hyperbola.[16] It is in fact an anonym composition related to the parabolic mirrors which deals in the conclusion with the hyperbola and the properties of its asymptotes.
From the thirteenth century onwards, Hebrew compositions begin to appear. This is the direct result of the publication of the *Guide* and its increasing influence. The Jewish authors, who didn't know the work of Apollonius, tried to give a direct demonstration of the geometrical property of the asymptotes of the hyperbola or discuss the philosophical aspects of the notion

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[12] Rashed, R. Al-Sijzi et Maimonide: commentaire mathématique et philosophique de la proposition II-14 des Coniques d'Apollonius, Archives internationales d'Histoire des Sciences, vol 37, n° 119, 1987, pp. 263 -296.
[13] Al-Tusi, Oeuvres Mathématiques. Algèbre et Géométrie au 12e siècle. Texte établi et traduit par R. Rashed, 2 vol, Paris 1986. Vol 1: pp. 5 – 15 and Vol 2: pp. 129 -130.
[14] See clagett (1954).
[15] A list of these manuscript was given by Freudenthal (2000) in *Maimonides and the Sciences*, Kluwer, 2000 p. 50.
[16] See Clagett (1980) chap IV: The speculi almukefi composition, anonymous composition of the 13 – 14th centuries. It was improved and developed by Regiomontanus and published by A. Gogava.

of infinity. They generally rested on former sources like the former Arabic compositions and from the 15th century onwards, also on Latin texts.

Tony Levi has examined from a mathematical point of view the different Hebrew texts, printed and still in manuscript, with the aim finding connections with the former texts in Arab and Latin. He identified texts (printed or in manuscript) authored by:

- Shimon Mottot, Italy 15th century. The only information extant is that this author was contemporary and befriended with the mathematician Mordechai Finzi of Mantua.
- Efodi alias Profiat Duran, the Provençal name of Moses ha-Levi from Perpignan (1345 – 1420), physician, astronomer and polemist against the Christian faith and the Jews believing in it. His commentary on the *Guide* is classic.
- Solomon ben Isaac.
- Eliahu ben Isaac ben Eliahu ben Aharon ha-Kohen (15-16th century).
- R. Moses ben Abraham Provençali (1503 – 1575) of Mantova.[17]

All these texts consider the section of a cone of revolution cut by a plane parallel to the axis of the cone.

Tsvi Langermann has also identified a manuscript[18] about the lines which never meet by Mordechai Finzi.[19] In this manuscript the considered curve is not a hyperbola but a conchoid. This curve had already been studied by the Greek mathematicians Nicomedes,[20] Eutocius[21] and Pappus.[22] He noted also that Finzi could have been inspired by another text published recently[23] and entitled '*Meyasher Aqov*' ascribed to a certain Alfonso, who could perhaps be identified with Abner of Burgos (1270 – 1340).[24]

Dr Shimon Bollag has gathered the titles of the Hebrew printed texts related to the mathematical aspects of the 'two lines'. Their aim was to find a direct prove of Maimonides' proposition.

1. ר' שם טוס פלקירא, פירוש למורה נבוכים, מורה המורה, פרסבורג 1837.
2. ר' יוסף ב"ר אבא מארי כספי, ב 'עמודי כסף' ו 'משכיות כסף', פרנקפורט 1848.
3. ר' משה בן יהושע הנרבוני, פירוש למורה נבוכים, וינה 1852.
4. ר' יצחק בן משה הלוי דוראן בעל האפודי, פירוש למורה נבוכים, ויניציה 1551.
5. ר' שמעון מוטוט, ביאור בסוף ספרו על האלגברה 1473 1445. כת"י.
6. ר' משה בן אברהם פרובינצאל, מאמר שני הקוים, בתוך מורה נבוכים, סביוניטה 1553.
7. ר' שלמה יוסף הרופא, יש"ר מקנדיא, ספר אלים, אמסטרדאם 1639. ראה ספר אלים, מעין חתום. עמ' תכ"ד תכ"ו בדפוס אודסה 1867.
8. ר' שמעון בן שמואל בכרך, שו"ת חוט השני, פרנקפורט 1679. (ס' קע"ד)
9. ר' יאיר חיים בו שמעון בכרך, שו"ת חוות יאיר, פרנקפורט 1699.

[17] He was one of the great rabbinic authorities of his time.

[18] Bodley. Mich: 35a 91b.

[19] Italian mathematician and scholar died in 1476. See Langermann (1988) pp. 33 – 38.

[20] Eutocius of Ascalon (about 480 – 540 C.E), Greek mathematician, author of commentaries on Archimedean treatises and on Apollonian conics.

[21] Nicomedes (about 280 – 210 B.C.E), ancient great mathematician who discovered the conchoids.

[22] Pappus of Alexandria (about 290 – 350 C.E), Greek mathematician and compiler author of an important theorem in geometry.

[23] Alfonso: *Meyasher Aqov* ed. Gluksina, G.M. Moscow, 1983. See Langermann (1999) pp. 33 – 39.

[24] Abner of Burgos (about 1270 – 1340) was an apostate and anti-Jewish polemist. He was converted at the age of 50 under the name of Alfonso de Valladolid.

10. ר' יהונתן בן יוסף מראזנאי, ספר ישועה בישראל , פרנקפורט 1720.
11. ר' שר שלום אביעד בן מנחם שמעון בזיליה, ספר אמונת חכמים, מנטובה 1730.
12. ר' עמנואל חי בן אברהם ריקי, ספר חושב מחשבות, אמסטרדאם 1732.
13. ר' יצחק בן משה סטאנוב, ספר אדר היקר, ברלין 1772.

Note that the manuscript[25] of R. Moses Provençal was ended already in 1549. It was immediately translated into Italian and edited by R. Joseph Shalit of Mantova in 1550. Cardano[26] drew on this work of R. Provençal in his 1554 enlarged treatment of the Problem. Peletier[27] who examined also this problem was influenced by the work of Cardano. Finally we note the important work of Francesco Barozzi[28] who published in 1586 a recapitulative book *'Admirandum illud geometricum problema tredecim modis demonstratum quod docat duas lineas in eodem plano designare, quae numquam invicem coincident, etiam si in infinitum protahantur, et quando longius producuntur, tanto sibiinuicem propiores euadant'* detailing thirteen different demonstrations of the theorem of the two lines, and among them the method of R. Provençal, the works of Cardano and Peletier.[29]

The mathematical approach of al- Sijzi.

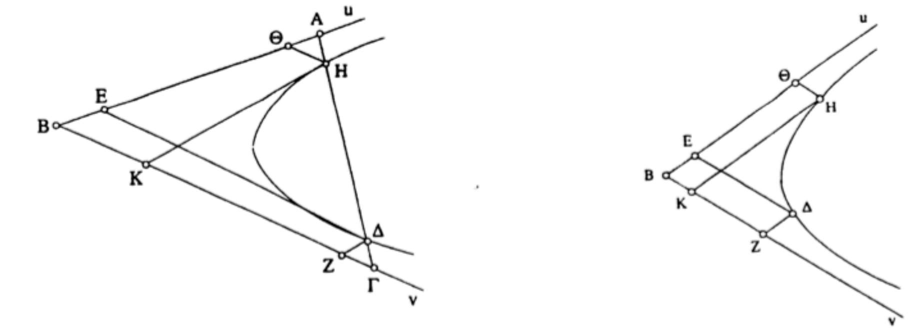

Figure 3: the demonstration of al-Sijzi. Left: the figure explaining the proposition II-10 of Apollonius. Right: The proposition II-10 with respectively HK, ΔZ and ΔE, Hθ parallel to the asymptotes.

It is not our intention to examine each mathematical approach. However, the solutions of the Arab mathematicians al-Sijzi and al-Tusi have a special importance because of their own merits, their influence on the later Latin and Jewish mathematicians and because they, if we exclude al-Tusi, were probably known from Maimonides. Al-Sijzi used the proposition II-12 of Apollonius. According to it, let Bu and Bv denote the asymptotes of a hyperbola. From the point Δ of the hyperbola, we draw to rights, the one intersecting Bu in E and the second

[25] Still extant.
[26] Mathematician and Mechanical engineer (1501 – 1576), known for his solution of the cubic equation, the Cardan shaft with universal joint and the Cardan ring.
[27] Mathematician and humanist poet (member of La Pleiade) (1517 – 1582).
[28] Mathematician and Astronomer (1537 – 1604). He published in 1586 his work summarizing the thirteen methods of demonstration of the problem of the asymptote. In 1587 he was accused of apostasy and heresy; he was severely finned.
[29] See Clagett (1980).

intersecting Bv in Z. From another point H of the hyperbola we draw the parallels to ΔE and Δz, they intersect Bu and Bv respectively in θ and K: we have then $\Delta E . \Delta Z = H\theta . HK$.
Al-Sijzi considered the case when the rights drawn from H and Δ are parallels to the asymptotes. We have then, considering the hyperbola in the system of coordinates Bu, Bv of its asymptotes: $x_\Delta . y_\Delta = x_H. y_H$.[30] If we multiply both terms by sin(uBv) we can interpret it as the equality of the area of all the parallelograms constructed parallel to the asymptotes from any point of the hyperbola. Al-Sijzi uses this result for the proof of the proposition II-14 of Apollonius.

- The area of a line is zero. Thus the point Δ is never on the asymptote.
- When x_Δ increases y_Δ diminishes and it becomes smaller than any small quantity ε as soon as $> S / y_{\Delta.,}\sin(uBv)$ where S is the constant area of the parallelograms.

The mathematical approach of al-Tusi.

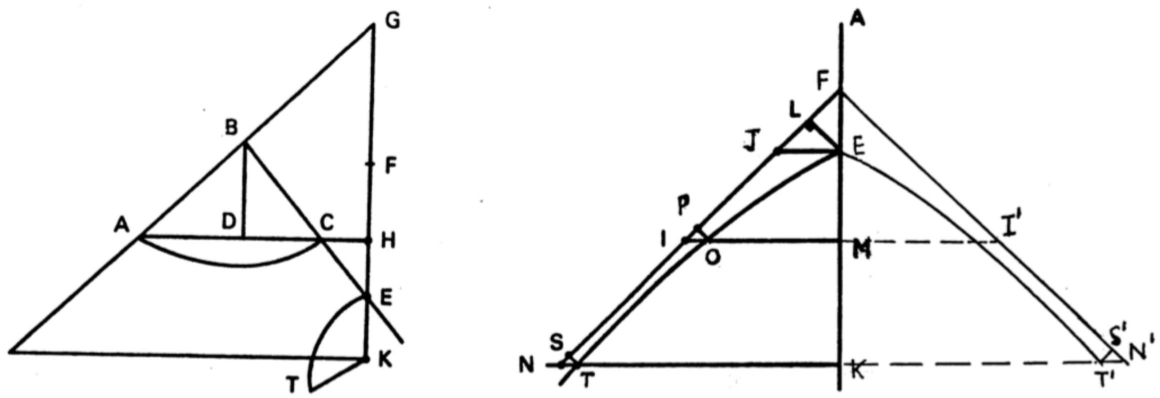

Figure 4: <u>left</u>. The triangle ABC is rectangle in B and isosceles, AB = BC. By rotation around BD, the axis of the cone, it generates a cone of revolution with φ = 45°. The cutting plane GFHEK cuts both nappes of the cone. E and H are the vertices of the hyperbola and F is its center. We will study the inferior branch TE. Right. We represent at a greater scale, the inferior branch of the hyperbola TET' and its asymptotes FS and FS'.

The approach of al-Tusi is less general than the former methods. He works with a cone with φ = 45°. Thus any section of the cone by a plane including the axis of the cone is a rectangular isosceles triangle.

1. Lemma.

Al- Tusi proves by application of the results of the proposition I-12 of the second book of Apollonius in this special case of an angle $\varphi = 45°$, that $(GE + EK) * EK = TK^2$. This relation is also valid in according the modern methods of the analytical geometry. The equation of the hyperbola with regard to its axes Fx_0y_0 is: $x_0^2/a^2 - y_0^2/b^2 = 1$. Or: $y_0^2 = (b^2 / a^2)*(x_0^2 - a^2)$. If we make a translation of the system of coordinates toward Exy, where E is the vertex of the hyperbola then $x_0 = a + x$ and $y_0 = y$. The equation of the hyperbola in the new system of

[30] This property is very important because it proves the equivalence between the conic sections and the conics defined as representing the graph of the equations of the second degree in x and y. Indeed the equation of an hyperbola, defined in this last way, is also x.y = constant when the asymptotes are taken as the axes of coordinates.

coordinates is $y^2 = (b^2 / a^2)*((x+a)^2 - a^2) = (b^2 / a^2)*(x^2 + 2ax+a^2 - a^2) = (b^2 / a^2) *x* (x + 2ax)$. Now in the case of perpendicular asymptotes we have a=b and $y^2 = x* (x + 2ax)$.

2. Demonstration of the proposition.

After the proof of this lemma, al-Tusi examines the problem of the asymptotes. On fig4 left, let ABC a rectangular isosceles triangle be the section of a cone by a plane to which the axis BD of the cone belongs. Let GFHEK denote a plane perpendicular to the former plane; it cuts the cone along a hyperbola: F is its center, G and E are the vertices and TE is a part of one of its branches. On fig 4 right, we represented the branch TET' of the hyperbola: F is the center, E is the vertex and NF and N'F are the asymptotes. From Apollonius' proposition II-10 we know that I'Q . OI = N'T . TN= constant. But I'M = MF, therefore I'O = FM + MO = FMO. We have thus: FMO . OI = FKT . TN = constant. We note that KF > MF and KT >MO because of the former lemma. Thus FKT > FMO and therefore TN < OI and TS < OP because OP/OI = TS/TN=$\frac{\sqrt{2}}{2}$.

NK > TK because NK = KE + EF = x + a and $(TK)^2 = x* (x + 2ax)$ Thus NT>TS>0 but NT and NS will become smaller than any infinitely small ε as soon as EK = x is sufficiently great.

This demonstration is not fundamentally different than that of al-Sijzi. It rests on the same proposition II-10, as the proof of Apollonius; it is less general because it postulates an opening angle of φ = 45° of the cone but it allows considering a distance of the point of the hyperbola to the asymptote, perpendicular to it.

Philosophical considerations about the properties of the asymptotes of the hyperbola.

When Maimonides introduced his example with the asymptotes of a hyperbola, he had no mathematical problem, the demonstrations of Apollonius or of the Arab mathematicians that he followed, were clear and convincing. The only purpose was philosophical. Many thinkers, Jewish and non-Jewish, quoted later this piece or referred to it.

Isaac ben Abraham ibn Latif (Provence, about 1210 – 1280).[31]

'The knowledge of counting bears upon sensible reality, but its object stretches further and further to infinity. So also the sciences of geometry, *al-handasah* in Arabic, bears upon the sensible which stretches further and further until it disappears from the eye, the end existing only in the intellect, just as an infinite march. *So are also two lines between which there is a certain distance at the outset and which may go forth in such a way that the further they go, this distance diminishes and they come nearer to one another , but without it ever being possible for them to meet even if they are drawn forth to infinity*,'

Levi ben Gershom (South of France, 1288 – 1344).

'Infinite increase is impossible inasmuch as magnitude is taken to be in absolute body, even if we do not postulate this to be a natural body. For body, whatever body it may be, is comprehended within the size of the world. Ibn Rushd therefore says that since in its existence line is not separate from matter, the geometer who postulates a line greater than the

[31] Sefer *Rav Pe'alim*, Lemberg 1885 p. 21r. Quoted in Freudenthal (2000) p. 56 note 70.

world postulates something wrong and false….And Ibn Rushd further says that the geometer need not at all postulate such a line, for the proof bearing on the long and on the short line is one and the same. According to my own opinion, however, occasionally the geometer must postulate such a line. For instance, when defining the parallel lines, he says that if protracted infinitely to either side they do not meet. Indeed, lines which do not meet when drawn to an extent equal to the size of the world are not necessarily parallel. This is self-evident. The same holds of what the geometer says of *two lines which, the further they go, come closer but whose meeting is impossible, even if they are drawn forth to infinity*.'[32]

Hasdai Crescas (Barcelona 1340 – 1410).

It has been demonstrated in the book on *Conic Sections* that it is possible for a distance infinitely to decrease and still never completely to disappear. *It is possible to assume, for instance, two lines, which, by how much farther they are extended, are brought so much nearer to each other and still will never meet, even if they are produced to infinity.* If in the case of decrease, there is always residual distance which does not disappear, a fortiori in the case of increase it should be possible for a distance, though infinitely increased, always to remain limited.[33]

Newton, Isaac (1643 – 1727): He is repeating a century-old argument. The most influential statement of this reasoning had already been formulated in the *Guide of the Perplexed*. Both accept the pre-eminence of the power of the reason upon the imagination and the common sense.

'I admit that an infinite number of things is difficult to conceive, and is therefore taken by many people as impossible: but there are many things concerning numbers and magnitudes which to men not learned in mathematics will appear paradoxical, and yet are entirely true. As that …two neighboring bodies (corpora) are always able to approach one another and yet never touch each other….For mathematicians know that …*the distance between hyperbolae and their asymptotes, when they are produced, always becomes smaller but never vanishes*.'[34]

Montaigne, Michel Eyquem de (1533 – 1592): Les Essais, Livre II, chap. 17, (original text – Old French). Montaigne uses the argument in the opposite direction and he draws an opposite conclusion.[35]

[32] Levi ben Gershom, Supercommentary on *Averroes' Intermediate Commentary on Aristotle's Physics* III,iii,5. Manuscript in Paris mentioned by Langermann and quoted by Freudenthal (2000) p. 56, note 71.
[33] Crescas critique of Aristotle, Wolfson, Harry A, Cambridge, Mass. 1929, p 207. Quoted in Freudenthal (2000) p. 52 note 72.
[34] McGuire J.E. 'Newton on Place, Time and God: An unpublished Source' in British Journal for the History of Science, 11 (1978), 114-29 at p. 119 (McGuire translation). Quoted in Freudenthal (2000), p. 52 note 1.
[35] Maimonides followed by Newton had opposed experience and demonstration against imagination and common sense and he had given pre-eminence to the firsts (see the two examples quoted in the *Guide*, I: 73). Montaigne mixes the arguments (he doesn't know the *Guide*) and he opposes now rational demonstration against the truth of the experience. By the tone of his prose and his appreciation of geometry, we see that he gives pre-eminence to the truth of the experience and uses the argument of the 'two lines' to prove the questionable character of the verbose and scholastic demonstrations versus the truth of the experience.

‘ Or ce sont des choses qui se choquent souvent ; et m’a l’on dit qu’en la Géométrie (qui pense avoir gagné le haut point de certitude parmy les sciences) il se trouve des démonstrations inévitables subvertissans[36] la vérité de l’expérience : comme Jacques Peletier[37] me disoit chez moy qu’il avoit trouvé *deux lignes s’acheminans l’une vers l’autre pour se joindre, qu’il vérifoit toutefois ne pouvoir jamais, jusques à l’infinité, arriver à se toucher*.’[38]

Voltaire (Arouet, François 1694 – 1798): Dialogues philosophiques VII, 1.

‘…N’êtes-vous pas forcé d’admettre *les asymptotes en géométrie sans comprendre comment ces lignes peuvent s’approcher toujours, et ne se toucher jamais*? N’y a-t-il pas des choses aussi incompréhensibles que démontrées dans les propriétés du cercle ? Concevoir donc qu’on doit admettre l’incompréhensible, quand l’existence de cet incompréhensible est prouvée.’[39]

The problematic of the two lines today.

We note that all the pieces mentioned and considered above date back the seventeenth century and exceptionally the eighteenth century for the last document. Apparently this problem is not anymore very much in the news. We can explain it by the following elements:

- The rabbis have lost any interest for mathematics and generally for philosophy.
- *The Guide of the Perplexes* is considered today with reservation by the orthodox community.
- Modern people, graduated from the secondary school education system (maturity), and certainly those who attended calculus courses at university and engineering schools, overcome their discomfort and perplexity of the infinite. They are accustomed to reasoning on the infinite and can calculate limits of convergent series and undetermined values of functions of the type: $\frac{0}{0}, 0 \times \infty \; or \; \frac{\infty}{\infty}$. In contradiction with the text of Levi ben Gershom quoted above, they accept even that two parallel rights, which are separated by a fixed distance, cut at the infinite: this allows generalizing many properties in a more homogenous presentation.
- Although we dispose today of the best texts, translations and commentaries, the *Guide of the Perplexed* is no more read by the well-read men, only specialists and exceptional people read it.
- Today the conics are defined as the curves having a function of the second degree as equation. The notion of asymptote lost its paradoxical character.

The hyperbola referred to its symmetry axes.

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[36] ‘Subvertissans’, today ‘subvertissant’ means subverting, reversing.
[37] Jacques Peletier du Mans (1517 – 1582) was a (today forgotten) mathematician and poet, befriended with Ronsard and du Bellay. He wrote an essay on the problem of ‘the two lines’ included in the book of Francesco Barozzi.
[38] Michel de Montaigne, Les Essais, ed. Alexandre Micha, Paris, 1969, pp. 236 – 7. Quoted by Rashed (2000) p.172, note 36 and Freudenthal (2000) p. 56, note 74.
[39] Quoted by Rashed (2000) p. 172, note 37.

The equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (1)

The equation of the asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $y = \pm \frac{b}{a} x$. (2)

The equation of the hyperbola can, in this particular case, be solved with regard to y

$a^2y^2 = b^2(x^2 - a^2)$ hence $\frac{y}{x} = \pm\frac{b}{a}\sqrt{1 - \frac{a^2}{x^2}}$. If we consider the part of the hyperbola situated in the first quadrant, we see that the hyperbola is always under the asymptote. It is only when x grows toward infinity that Lim $\frac{y}{x} = \pm\frac{b}{a}$.

The equation of the tangent to the hyperbola in the point x_1 y_1 is given by: $\frac{x.x1}{a^2} - \frac{y.y1}{b^2} = 1$.

It can also be written on the form: $\frac{y.y1}{x.x1} = \frac{b^2}{a^2} + \frac{b^2}{x.x1}$ where x and y represents the current point of the tangent, x_1 y_1 is the chosen point and point of contact. If this point of contact tends toward infinity then $\frac{y1}{x1}$ tends toward $\pm\frac{b}{a}$. The equation of the tangent tends toward $\frac{y}{x} = \pm\frac{b}{a}$. The tangent of the hyperbola tends toward the asymptote when the point of contact tends to infinity.

The conic sections compared to the curves defined by functions of the second degree.

The Greeks defined the conics, as the word indicates it, as the sections of cones presenting a symmetry of revolution. This definition and conception remained valid until the 17th century. The analytic geometry was created and developed in the 17th century. It reached all its might thanks to the development of calculus in the 18th century. Today we define the conics through the functions of the second degree and their focus-directrix property. We can define, materialize and draw with precision the conics, the tangents and the asymptotes. The problem is then to establish and prove on an indisputable manner that there is an identity between the two approaches, and that the properties established by the methods of the analytical geometry are still valid for the conic sections. Normally this aspect of the problem is generally not raised during the scholar curriculum.

1. **Analytical method.**

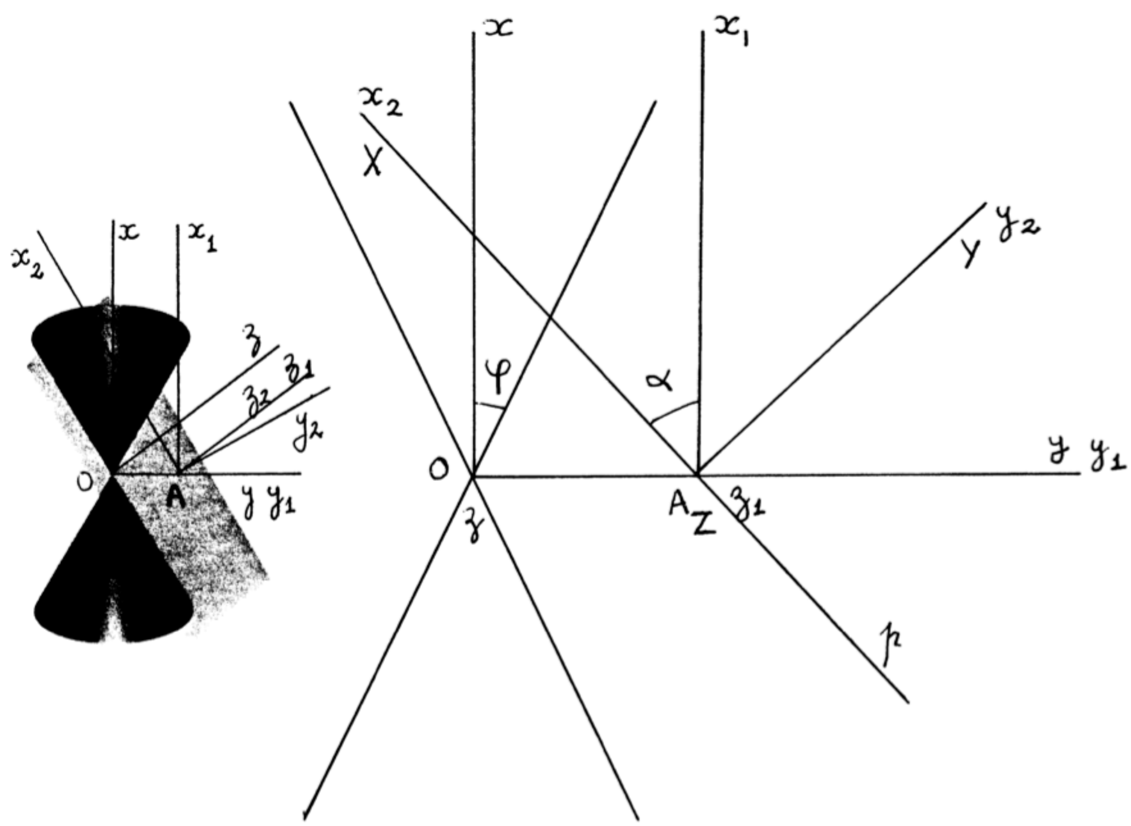

Figure 5: Intersection of a cone by a cutting plane. Left: perspective with representation of the cone, the plane perpendicular to plane xy and representation of the three system of coordinates Oxyz, $Ax_1y_1z_1$ and $Ax_2y_2z_2$ also called AXYZ. A is the intersection of the cutting plane p with the axis Oy. It is unfortunately difficult to distinguish on the figure. Right: Projection of the precedent figure on the frontal plane xy. The point O represents also the axis z, the point A represents also the axes z_1 and z_2 which we rename Z.

Let us consider a cone of revolution in a system of rectangular coordinates Oxyz. The origin O coincides with the vertex of the cone and the axis x with the symmetry axis of the cone. The cone is cut by a plane p perpendicular to plane Oxy and parallel to Oz and oblique with regard to Ox and Oy. Let us call the point of intersection of this plane with Oy, A. We consider a second system of coordinates $Ax_1y_1z_1$ parallel to Oxyz, where A is the point of intersection of plane p by Oy and a third system of coordinates $Ax_2y_2z_2$ which is the result of the rotation of $Ax_1y_1z_1$ around axis z_1 so that axis y_2 becomes perpendicular to the plane p.

Equation of the cone in the axes Oxyz.

We consider a cone of revolution around the axis Ox, where O is its vertex we call φ the angle between the axis and each element of the cone. Any plane parallel to the plane Oyz and distant from it by x cuts the cone along a circle of radius x tgφ. The equation of the circle is $y^2 + z^2 = r^2$ where r = x. tgφ. The equation of the cone is thus $y^2 + z^2 = x^2 \, tg^2\varphi$. (1)

Changing the system of coordinates Oxyz into $Ax_1y_1z_1$ by a translation of (a) parallel to axis Oy. We have $x = x_1$, $y = y_1 + a$, $z = z_1$. The equation (1) becomes $(y_1+a)^2 + z_1^2 = x_1^2.tg^2\varphi$ or $y_1^2 + 2ay_1 + a^2 + z_1^2 = x_1^2 \, tg^2\varphi$. (2)

Changing the system of coordinates $Ax_1y_1z_1$ into $Ax_2y_2z_2$ by a rotation of angle α around axis Az_1 where α is the angle between $Ax_1$and Ax_2. The formulas of the change of coordinates are the classical formulas of a system of rectangular coordinates rotating by an angle α[40] after permutation of x_1 and y_1, x_2 and y_2.

$x_1 = x_2 \cos\alpha + y_2 \sin\alpha$ and $y_1 = -x_2 \sin\alpha + y_2 \cos\alpha$. Formula (2) becomes then
$(- x_2 \sin\alpha + y_2 \cos\alpha)^2 + 2a(- x_2 \sin\alpha + y_2 \cos\alpha) + a^2+ z_2^2 = (x_2 \cos\alpha + y_2 \sin\alpha)^2.tg^2\varphi$. (3). The system of coordinates is the final system, we call it AXYZ and rewrite (3)
$(- X \sin\alpha + Y \cos\alpha)^2 + 2a(- X \sin\alpha + Y \cos\alpha) + a^2+ z_2^2 = (X \cos\alpha + Y \sin\alpha)^2.tg^2\varphi$. (4)
$X^2\sin^2\alpha+Y^2\cos^2\alpha - 2XY\sin\alpha.\cos\alpha - 2aX\sin\alpha+2aY.\cos\alpha+a^2+Z^2= (X^2\cos^2\alpha+2XY\sin\alpha.\cos\alpha+$
$+Y^2 \sin^2\alpha) tg^2\varphi$. Or after reorganization:
$X^2 (\sin^2\alpha - \cos^2\alpha\, tg^2\varphi) +Y^2 (\cos^2\alpha - \sin^2\alpha\, tg^2\varphi) -2XY\sin\alpha.\cos\alpha (1 + tg^2\varphi) - 2aX.\sin\alpha +$
$+2aY.\cos\alpha+a^2+Z^2 = 0$. (5)
The equation of the cutting plane p in the system of coordinates AXYZ is Y = 0. The equation of the conic section is then in the axes AXZ:

$\mathbf{X^2 (\sin^2\alpha - \cos^2\alpha\, tg^2\varphi) - 2aX\sin\alpha +Z^2 +a^2 = 0.}$ (6)

The equation of a conic in axes XY is: $f(X,Y) = AX^2 + 2BXY + CY^2+ 2DX +2EY + F = 0$. We can divide them according to the value of $AC - B^2$ depending on whether it is >, < or = 0. In our case $AC - B^2 = (\sin^2\alpha - \cos^2\alpha\, tg^2\varphi) = (1 - \cos^2\alpha - \cos^2\alpha\, tg^2\varphi) = 1 - \cos^2\alpha\, (1+tg^2\varphi) =$
$= 1 - \cos^2\alpha / \cos^2\varphi$ because $\cos^2\varphi = 1 / (1+ tg^2\varphi)$.

$\mathbf{X^2 (1- \cos^2\alpha / \cos^2\varphi) - 2aX\sin\alpha +Z^2 +a^2 = 0.}$ (6')

If $AC - B^2 = 1 - \cos^2\alpha / \cos^2\varphi > 0$ the conic is an ellipse. This is the case if α > φ, we have then cosα < cosφ and indeed then plane p cuts one nappe of the cone.

If $AC - B^2 = 1 - \cos^2\alpha / \cos^2\varphi < 0$ the conic is a hyperbola. This is the case if α < φ, we have then cosα > cosφ and indeed then plane p cuts both nappes of the cone.

If $AC - B^2 = 1 - \cos^2\alpha / \cos^2\varphi = 0$ the conic is a parabola. This is the case if α = φ, we have then cosα = cosφ and indeed then plane p cuts one nappe of the cone and is parallel to one element of the cone. The equation of the parabola is then:

$\mathbf{- 2aX\sin\alpha +Z^2 +a^2 = 0.}$ (6'') and the abscissa of the vertex, for Z = 0 is $\mathbf{X = a / 2 \sin\alpha}$ (7)

The axis X is one symmetry axis of the conic but A is not the center of the conic. The abscissa of the center is given by the equation f'(X) = 0

Thus: $2X (1 - \cos^2\alpha / \cos^2\varphi) - 2a \sin\alpha = 0$ $\mathbf{X =a \sin\alpha / (1 - \cos^2\alpha / \cos^2\varphi)}$. (8)

If α = 0 the cutting plane is parallel to the axis of the cone and the hyperbola is symmetric with regard to point A, which is the center of the hyperbola. In our equation (6) the coefficient of X^2 becomes $1 - 1 / \cos^2\varphi = (\cos^2\varphi - 1)/ \cos^2\varphi = - tg^2\varphi$.

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[40] The angle α is < 90°. If α = 0°. The cutting plane is perpendicular to the axis of the cone, the section is circular. The cutting plane the not cut the axis Ox and there is no point A.

$\mathbf{tg^2\varphi\ X^2 - Z^2 = a^2}$ **or** $\frac{X^2}{\frac{a^2}{(tg\ \varphi)^2}} - \frac{Z^2}{a^2} = \mathbf{1}$. The half of the axes are a / tg φ and a.

If $\alpha > \varphi$, the conic is an ellipse. The equation (6) is now $(1 - \cos^2\alpha / \cos^2\varphi) X^2 - 2aX\sin\alpha + Z^2 = a^2$ and the equation (8) shows that the abscissa of the center is positive: the center is above point A.

If $\alpha < \varphi$, the conic is a hyperbola. The equation (6) is now $ABS(1 - \cos^2\alpha / \cos^2\varphi) X^2 + 2aX\sin\alpha - Z^2 = a^2$ and the equation (8) shows that the abscissa of the center is negative: the center is beneath point A. The asymptotes of the hyperbola are given by
$ABS(1 - \cos^2\alpha / \cos^2\varphi) X^2 - Z^2 = 0$ (9)

or $ABS(1 - \cos^2\alpha / \cos^2\varphi) X - Z = 0$ (10a)
and $ABS(1 - \cos^2\alpha / \cos^2\varphi) X + Z = 0$. (10b)

Now equation (9) is also the equation of the intersection of the cone by the plane p when a=0. Thus the family of the hyperboles which are the intersection of the cone with the plane p(a) have the same asymptotes; they are the intersection of the cone with the plane p when a = 0.

We have found all the parameters of the conic in function of the angle φ between the axis of the cone and its elements and the angle α between the axis of the cone and the cutting plane.

The present proof is general and complete. It proves definitively the equivalence between the conic sections and the curves of the second degree. Therefore all the properties of the curves of the second degree found by the methods of the analytical geometry, especially the focus-directrix properties are valid for the conic sections. I did not find such a proof in the many books I consulted but this elementary proof is certainly not new.

2. Geometrical method.

This method is based on the discovery in 1822 of the spheres of Dandelin,[41] name given to the two spheres tangent to the cone and to the cutting plane. In the case of an elliptic section the two spheres of Dandelin are on both sides of the cutting plane. In the case of a hyperbolic section, the two spheres are on the same side of this cutting plane and in the case of a parabolic section there is only one sphere of Dandelin. On this manner Dandelin discovered a surprising way of finding the focus and directrix by constructing these two spheres that, each of them, touches the cone along a circle and the cutting plane of the cone in a single point. The intersection of the plane and the cone is thus a conic section and the point at which either sphere touches the plane is a focus of the conic section. The intersection of the cutting plane

[41] Belgian Engineer (1794 - 1847) of French origin, graduated from the famous "Ecole Polytechnique", he taught at the "Ecole des Mines" of Liège. He was the author of studies on the conics (1822 – 1827). According to some, the theorems of Dandelin were the result of his collaboration with Quetelet (1796 - 1874). Some call the theorems of Dandelin: the Belgian theorems.

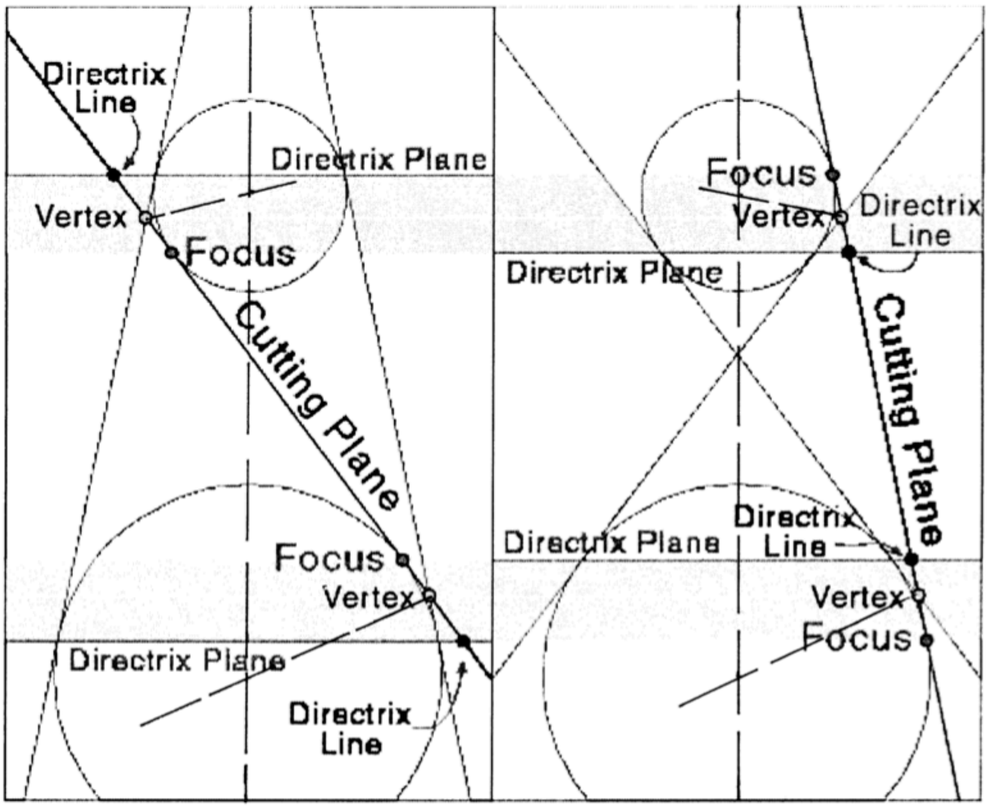

Figure 6: Construction of the Spheres of Dandelin. We project the cone and the cutting plane on a plane containing the cone's axis and perpendicular to the cutting plane. We want to find the center of the circles tangent to the cone and the cutting plane. The center is at the intersection of the vertical axis and on the bisecting line of the angle between the cutting plane and the section of the cone. On the left figure, both spheres are in the same nappe of the cone and on both sides of the cutting plane. On the right figure both spheres are in the two nappes of the cone, on the same side of the cutting plane. If the cutting plane is parallel to the section of the cone, there is only one sphere of Dandelin.

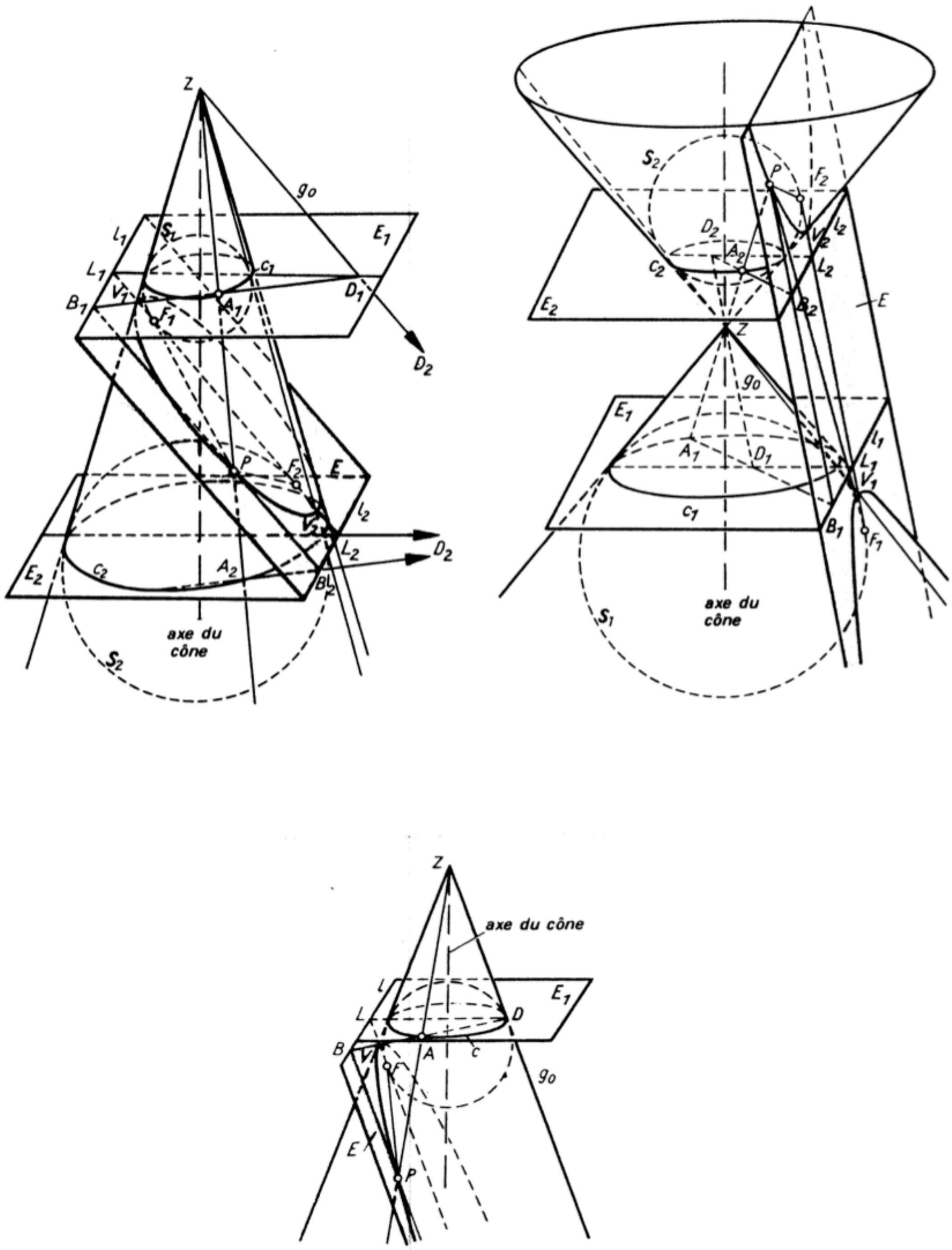

Figure 7: The spheres of Dandelin in the tree possible cases: above left, the section is an ellipse, above right the section is a hyperbola and below the section is a parabola. On these figures, V_1 and V_2 are the vertices of the section. F_1 and F_2 are the point of contact of the spheres of Dandelin with the cutting plane and the foci of the conic section. P is a random point of the section. C_1 and C_2 are the circles of contact of the spheres of Dandelin with the cone. PA_1A_2 is an element of the cone crossing C_1 in A_1 and C_2 in A_2. We note that $PF_1 = PA_1$ and $PF_2 = PA_2$. In the case of the ellipse $PF_1 + PF_2 = A_1A_2 =$ constant. In the case of the hyperbola $PF_1 – PF_2 = A_1A_2 =$ constant. The intersection of the planes E_1 and E_2 of the circles of contact C_1 and C_2 with the cutting plane are the straight lines l_1 and l_2, the directrices of the conic. The straight lines V_1V_2 is the main axis of the conic, g_0 is parallel to it but we don't use it. B_1 and B_2 (B in the case of the parabola) are the feet of the perpendicular drawn from P on the directrices. $PF_1 / PB_1 = PF_2 / PB_2 =$ constant.

with the plane of the circle of contact of each sphere of Dandelin is a directrix of the conic.[42] The two circles of contact are parallel and the two directrices (ellipse and hyperbola) are parallel.

First theorem of Dandelin.

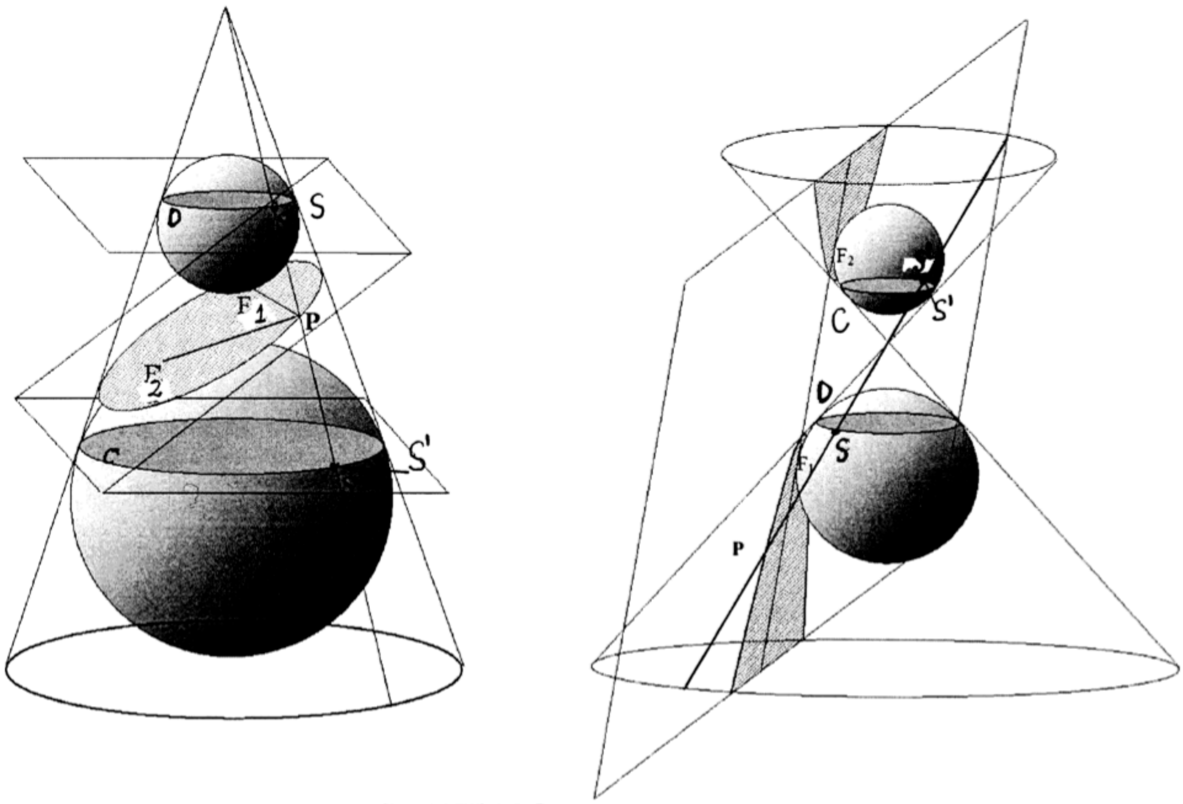

Figure 8: The first theorem of Dandelin. The two figures are simplified and more readable. F_1 and F_2 are the points of contact of the spheres with the cutting plane. C and D are the circles of contact of the two spheres with the cone. P is a point of the conic section. S and S 'are the points of intersection of an element of the cone passing through P with the circles of contact C and D. PS = PF_1 (tangents to the little sphere) and PS' = PF_2 (tangents to the greater sphere). In the left figure $PF_1 + PF_2$ = SS' the constant distance between the planes of C and D, perpendicular to the axis of the cone, measured along an element of the cone. In the right figure $PF_2 - PF_1$ = SS' the constant distance between the planes of C and D, perpendicular to the axis of the cone, measured along an element of the cone.

- An elliptic section of a cone is the locus of points such that the sum of their distances to two fixed points, the points of contact of the cutting plane with the spheres of Dandelin, is constant.[43]

[42]Some of these historical elements as well as the demonstration are already published in Morton (1830) p. 228.
[43] In fact it seems that Pappus was already aware of the properties of the foci but the theorem of Dandelin makes this demonstration easier and gives the geometrical meaning of their position. In fact if Pappus knew the foci and

- A hyperbolic section of a cone is the locus of points such that the difference of their distances to two fixed points, the points of contact of the cutting plane with the spheres of Dandelin, is constant.

Consider the illustrations depicting a plane intersecting a cone to form an ellipse or a hyperbole. The two Dandelin spheres are shown. The circles of contact of these spheres with the cone are the circles C and D. Each sphere touches the plane at a point of contact F_1 and F_2. Let P be a typical point of the conic, in the case of the ellipse we must prove that the sum of the distances PF_1 +P F_2 is a constant. In the case of the hyperbola we must prove, if P belongs to the branch 1 of the hyperbola, that the difference of the distances $PF_2 - PF_1$ is a constant.

A line passing through P and the vertex of the cone intersects the two circles of contact at points S and S'. As P moves along the conic, S and S' move along the two circles. Now the distance PF_1 is the same as the distance PS because lines PF_1 and PS are both tangent to the same sphere. Likewise the distance PF_2 is the same as the distance PS' because lines PF_2 and PS' are both tangent to the same sphere. Consequently the sum $PF_1 + PF_2$ in the case of the ellipse or the difference $PF_2 - PF_1$ in the case of the hyperbola, remains constant as P moves along the conic because and is equal to the distance SS' between two parallel planes perpendicular to the axis of the cone, along an element of the cone of revolution, making a constant angle with it.

Second theorem of Dandelin.

A conic section of a cone is the locus of points for which the distance from a focus is proportional to the distance from the corresponding directrix. The constant ratio between the distance to the focus and the distance to the directrix is denoted by'**e**' the eccentricity of the conic section. According to the value of the eccentricity:

- $e < 1$: ellipse and $e = 0$: circle
- $e = 1$: parabola.
- $e > 1$: hyperbola.

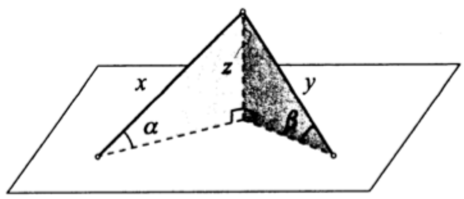

Figure 9: The ratio of the length of two segments x and y from a point to a plane $x/y = \sin\beta/\sin\alpha$.

their properties, then necessarily there was an equivalence between the conic sections and the graphs of the equation of the second degree in x and y, f (x,y) = 0, which have indeed the same properties.

Lemma. The lengths of any two line segments from a point to a plane are inversely proportional to the sines of the angles that the line segments make with the plane.

Indeed we observe on fig that z = x sin α = y sin β, hence **x / y = sin β / sin α.**[44]

Proof. Let d denote the line of intersection of p and q and let B denote the foot of the line segment from P perpendicular to the line d. On fig 10 left and fig 10 right we represent one nappe of the cone: p is the cutting plane. On fig 1o left plane p cuts one nappe of the cone along an ellipse, on fig 10 right plane p cuts the cone along a parabola or a hyperbola (the drawing is limit between the two cases). C is the circle of contact of the cone and the represented sphere of Dandelin, S is the point of intersection of this circle with the element of the cone (a line of the cone passing through the vertex) passing through P.

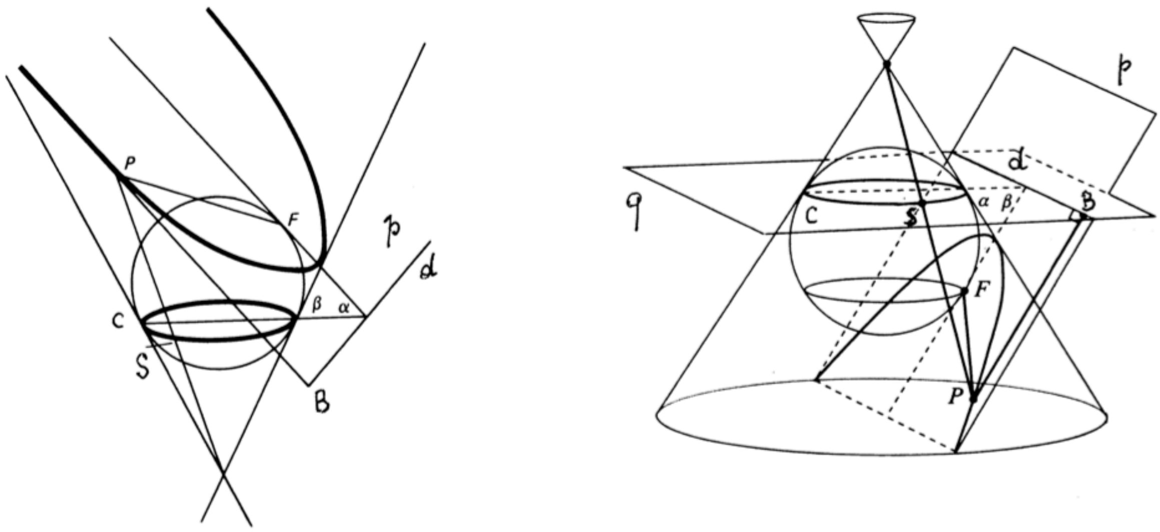

Figure 10: The plane p cuts a cone. F is the point of contact of the cutting plane and the sphere of Dandelin considered, d is the intersection of the cutting plane p and q, the plane of the circle of contact of the sphere of Dandelin. B is the foot of the perpendicular drawn from P on d. Then PF / PB = PS / PB = sin β / sin α. On the left figure the plane p cuts one nappe of the cone and the section is an ellipse. On the right figure the situation is limit: the plane seems parallel to an element of the cone and the intersection is a parabola but if the plane was slightly more inclined the intersection would be a branch of hyperbola.

Then PF = PS (tangents to the sphere). Let α denote the angle that every element of the cone makes with q and let β denote the angle between q and p. Then PF / PB = PS / PB = sin β / sin α = constant.

The point F in the proof is a focus of the conic section and the line ‘d’ is the corresponding directrix. The constant is denoted by ‘e’, the eccentricity of the conic section. When p is parallel to one and only one element of the cone then α = β and e = 1 and the conic is a

[44] Note that the angles used in this theorem are different than those used in the analytical demonstration. In the analytical method φ is the half angle of opening of the cone and α is the angle between the axis Ax_1 and the cutting plane. In the geometrical method α is the angle between the elements of the cone and any plane perpendicular to the axis of the cone, thus 90° - φ, and β is the angle between the same plane and the cutting plane, thus β = 90° - α.

parabola. When p cuts every element of one nappe of the cone $\alpha > \beta$ and $e < 1$ and the conic is an ellipse. When p cuts both nappes of the cone, $\alpha < \beta$, $e > 1$ and the conic is a hyperbola.

Conclusion.

The connection between the subject of the two lines, the curve and its asymptote, with the Jewish culture is surprising. In fact the proposition II-14 of the second book of Apollonius, his Chef d'Oeuvre, has puzzled mathematicians and philosophers all through the history, until the beginning of modern times. Maimonides, in his quality of thinker, philosopher and distinguished mathematician did not remain insensitive to the question and he used it as an argument against the Calam at the end of the first part of his *Guide of the Perplexes*. This was sufficient to stimulate the interest of Jewish intellectuals, rabbis and philosophers with a mathematical interest: During the period 12th – 17th centuries they studied thoroughly the problem and tried, with the help of the extant literature, to find complete proves which did not require reporting to rare books written in Latin or Arabic. It is difficult to adduce a direct influence of Maimonides on gentile thinkers, as those quoted above, especially Voltaire,[45] but because of the interaction of the cultures and the deep influence of Maimonides' *Guide* on the Scholastic, it is certain that he had certainly an indirect influence. It should be noted that the *Guide* had been translated into Latin as early as about 1240[46] and was thus available to gentile thinkers. Moreover the three quotations above from Gentiles authors refer explicitly to Maimonides' argument that *the experienced or demonstrated reality can go beyond the imagination*. Anyhow the subject of the 'two lines' belongs now to the history of the Jewish thought. We have seen through the analysis of different proves of the proposition about the asymptotes, given by the ancients that the approach 'conic sections' and 'analytic geometry' are equivalent. The inventors of the analytic geometry, Descartes and Euler, still new the works of the ancients and were aware of this equivalence. This must explain why they did never raise the problem. For modern students and mathematicians, without knowledge of history of mathematics, the problem is serious and no satisfactory answer is proposed; for this reason we have proposed two direct proves. Both are instructive, especially the second, dating from the beginning of the 19th century and not widely known; it is of a rare elegance and it gives a geometrical understanding of the position of the foci and the directrices of a conic.

In 1984 dr. S Bollag wrote a short paper about the subject of the 'two lines' devoted to Jewish references about the subject and to the commentary of R. Moses Provençal.[47] Since that time not less than five papers, related to this subject and devoted to the study of the different compositions, the different manuscripts and their connections, were issued, proving that even if the subject is not more in the news, it still interests the historian of mathematics and science. Unless the discovery of new manuscripts, it seems that the subject is exhausted, the time has come to making a synthesis and finding back the original understanding of the first Greek and Arab mathematicians.

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[45] An outspoken 'antisemite'.

[46] Freudenthal (2000) p. 45.

[47] The paper includes a copy of the original and the translation in formulas of different passages, improving the understanding. The comparison of these formulas with those proposed by Werner (1522) p. 26 and summarized in Coolidge (1945), pp. 26 – 27, proves that R. Provençal certainly used his book. Note that he did not claim originality and recognized loans.

Appendix:

1. The conchoids.

Some scholars, namely Mordekhai Finzi and Solomon Delmedigo used the conchoids for defining the properties of the asymptote as expounded by Maimonides. Let us consider in a system of Cartesian coordinates a straight line passing through the vertex O and intersecting the straight line x = a, parallel to the y axis in a point Q. The two points P and P' of the right OP, on both sides of the right OP at a distance PQ = P'Q = c generate a conchoids when the straight line OQ moves. The form of the conchoids depends on the ratio c /a. When c = a, O is a turning point. When c > a it is a double point. The parallel x = a, is always an asymptote of both branches. If φ is the angle between OP and the axis Ox then QP = QP' = a / cos φ. The polar equation of the conchoids is then $r = \frac{a}{\cos \varphi} \pm c$ and x = r cos φ = $a \pm c \cos \varphi$.

$$y = r \sin \varphi = a\, tg\, \varphi \pm c \sin \varphi.$$

The asymptote of both branches is x = a.

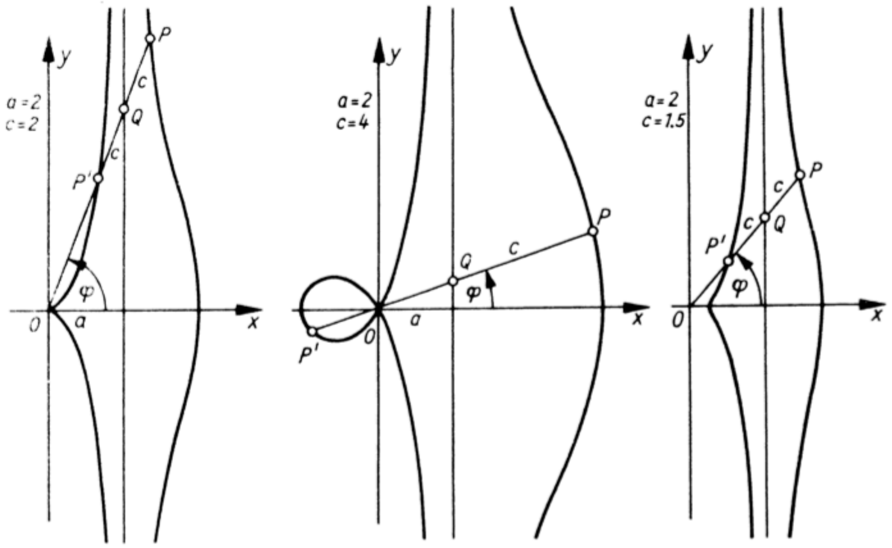


Figure 11: The Conchoids. Three different forms according to the value of c with regard to a. Left: c = a. Middle: c > a. Right: c < a.

2. The equivalence between the conic sections and the curves defined by functions of the second degree.

The equivalence between the approach of the conics by the methods of the analytic geometry and the focus-directrix approach is a well-established fact taught in secondary schools before the maturity. We demonstrated above the equivalence of these two approaches with the conic section approach and we noted that this aspect of the problem is practically and nearly always ignored in the modern mathematic scholar curriculum. I was puzzled by the fact that this problem had never been raised at the beginning of the development of the calculus and the analytic geometry. The examination of the book of Apollonius at the occasion of the understanding of the problems of the 'two lines' gave me a very simple answer to this interrogation. The propositions I-11, I-12 and I-13 of Apollonius' first book of conics give in an *endless* statement the evaluation of the area of the square constructed on the ordinate of the points of conics whose axis of symmetry is the axis Ax.[48] Thus if the adopted system of rectangular coordinates Axy is such that A is a vertex of the conic, Ay is tangent in A to the conic and Ax is its principal axis, the three above endless propositions can be summarized by the three following relations: I-11: $y^2 = 2\,\frac{b^2}{a}\,x = 2\,p\,x.$ Parabola.

I-12: $y^2 = 2\,\frac{b^2}{a}\,x - \frac{b^2}{a^2}x^2 = 2\,p\,x - \frac{p}{a}x^2$. Ellipse.

I-13: $y^2 = 2\,\frac{b^2}{a}\,x + \frac{b^2}{a^2}x^2 = 2\,p\,x + \frac{p}{a}x^2$. Hyperbola.

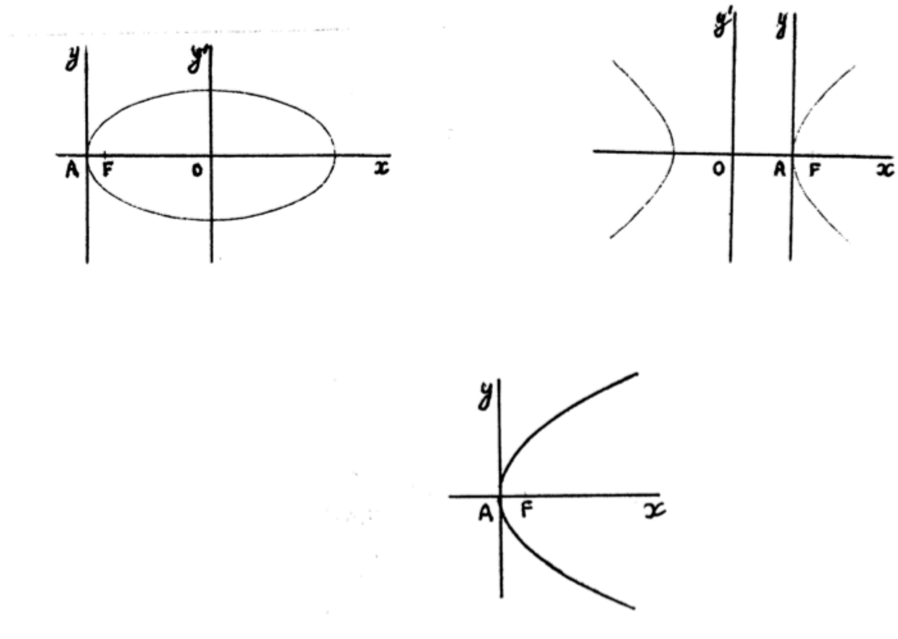


Figure 12: The conics with regard of the axes Axy, where A is a vertex of the conic and F is the focus. In this system of coordinates the equations of the conics are reduced to the formulas I-11, I-12 and I-13 above. In the ellipse AF = e and $e^2 = a^2 - b^2$. In the hyperbola AF = e and $e^2 = a^2 + b^2$. In the parabola AF = p/2.

We ascertain[49] that these relations are exactly the equations of the conics when the origin of the system of rectangular coordinates is shifted from the center of the conic to a vertex. It appears that any well-read mathematician[50] at the inception of the analytical geometry must

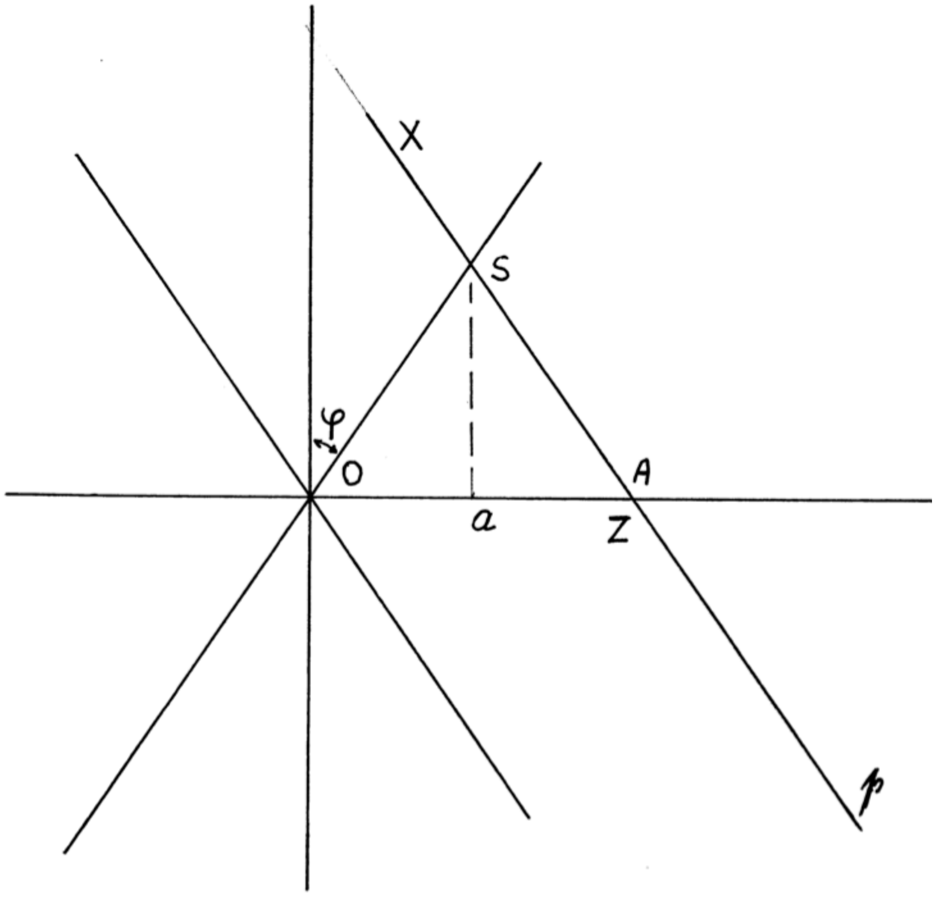
[48] See Ver Eecke (1923) pp. 21 – 31 and Fladt (1965), pp. 15 – 17.
[49] In any textbook of analytic geometry.
[50] As it was the case for the mathematicians of the 17th and 18th centuries. Don't forget that Halley, the great astronomer and close associate of Newton, knew Greek, Latin and Hebrew when he entered Queens college at Oxford (see Fried (2011) p. 5) and besides his scientific and astronomical achievements, he found the time in 1710, while professor of geometry at Oxford, to produce an edition of the Greek text of the conics of Apollonius'

know, from the text of Apollonius that the conic of Apollonius, those generated by conic sections, have the same equations than the conics considered in analytic geometry. In other words, there was no problem at all, from the beginning, at least for well-read mathematicians. Moreover these relations explain the denominations of the conics. Indeed Ellipse refers to the fact that the square constructed on y is less than 2px while hyperbola refers to the fact that this square is more than 2px.[51] Nevertheless Merzbah (2010) assigns to Fermat[52] the establishment of the correspondence of the conics with the general quadratic equations and the possibility to reduce the general quadratic equation of a conic into a simplified form through transformation of the system of coordinates (translation and rotation), Merzbach (2010) p. 323 and Fladt (1965) p. 64.

3. Geometrical interpretation of the formulas (7) above.

The vertex of the parabola is in S. We see indeed on fig. 13 that SA = a / 2 sinα = a / 2 sinφ.

Figure 13: the plane p is parallel to an element of the cone; the intersection is a parabola of vertex S.



Books I-IV, a translation into Latin from the Arabic version of Books V-VII and above all, a reconstruction of Book VIII! Nevertheless see note 52.

.[51] See Heath (1896) p. 9 and Coolidge (1945) p. 4. There are other explanations, however Ver Eecke (1923) p. 25 considers this explanation "subtle and likely"

[52] Fermat, Pierre de, French mathematician (1601 – 1665). Besides his achievements in arithmetic and number theory, Merzbach and Boyer consider that Fermat could well have discovered the first, as soon as 1629 the basis of analytic geometry and even of calculus, see Merzbach (2010) pp. 321 – 323. The two referred works were published in 1636: "Méthode pour la recherché du minimum et du maximum" and "Ad locos planos et solidos isagoge".

4. Geometrical interpretation of the formulas (8) and above.

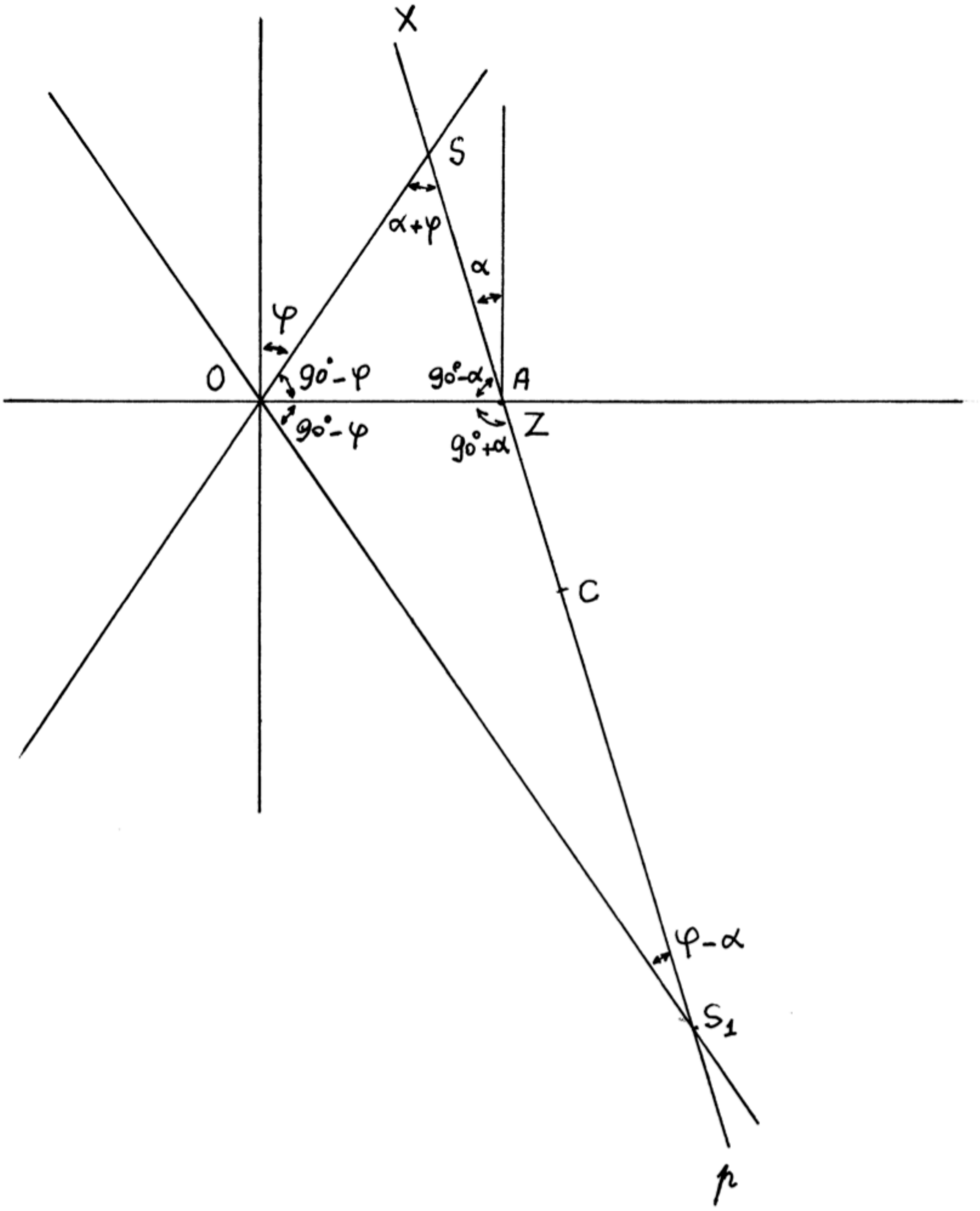


Figure 12intersects both nappes of the cone. S and S1 are the vertices of the hyperbola and C is its center.

On fig 14, $\varphi > \alpha$, and the intersection is a hyperbola. S and S1 are the vertices, M is the center. The theorem of the sines gives $SA = a \cos\varphi / \sin(\alpha+\varphi)$, $AS_1 = a \cos\varphi / \sin(\varphi-\alpha)$, $OS = a \cos\alpha / \sin(\alpha+\varphi)$, $OS_1 = a \cos\alpha / \sin(\varphi-\alpha)$ and $SS1 = a \cos\alpha \sin2\varphi / \sin(\alpha+\varphi) \sin(\varphi-\alpha)$.

$$AC = SC - SA = (SS1 / 2) - SA = \frac{2a \cos\alpha \sin 2\varphi}{2\sin(\alpha+\varphi)\sin(\varphi-\alpha)} - \frac{2\,a\cos(\varphi)\sin(\varphi-\alpha)}{2\sin(\alpha+\varphi)\sin(\varphi-\alpha)}.$$

Now sin(φ-α)= sinφ cosα – cosφ sinα and sin(φ+α)= sinφ cosα + cosφ sinα.

Numerator: 2a cosα sinφ cosφ – 2a cosφ (sinφ cosα – cosφ sinα) = 2a $\cos^2$φ sinα.

Denominator: 2(sinφ cosα + cosφ sinα) (sinφ cosα – cosφ sinα) =
2($\sin^2$φ $\cos^2$α – $\cos^2$φ $\sin^2$α) = 2 ($\sin^2$φ $\cos^2$α – $\cos^2$φ (1 – $\cos^2$α)) =
2($\sin^2$φ $\cos^2$α + $\cos^2$φ $\cos^2$α – $\cos^2$φ) = $\cos^2$α – $\cos^2$φ.

Finally AC = $\frac{a\,cos^2\varphi\,sin\alpha}{cos2\alpha - cos^2\varphi} = \frac{a\,sin\alpha}{\frac{cos^2\alpha}{cos^2\varphi} - 1}$.

In our case φ > α and cosφ < cosα and AM is negative i.e. in the negative direction of the axis AX, hence AC = $\frac{a\,sin\alpha}{1 - \frac{cos^2\alpha}{cos^2\varphi}}$ as in formula (8).

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